A certain algebraic construction of quasicrystals and their isomorphism classes

2000 J. Phys. A: Math. Gen. 33 8483

(http://iopscience.iop.org/0305-4470/33/47/311)

J. Phys. A: Math. Gen. **33** (2000) 8483–8487. Printed in the UK PII: S0305-4470(00)11998-5

A certain algebraic construction of quasicrystals and their isomorphism classes

Tatsuya Kimijima and Jun Morita
Institute of Mathematics, University of Tsukuba, Tsukuba, Ibaraki 305-8571, Japan

Received 21 February 2000, in final form 9 May 2000

Abstract. Certain quasicrystals will be realized in cyclotomic fields, and their isomorphism structures will be given in the case of seven-fold or 30-fold symmetry.

Let n be a natural number greater than 2, and $\varphi(n)$ the Euler function. Then $\varphi(n)$ is even, and we write $\varphi(n) = 2m$. We take a primitive nth root of unity, say $\zeta = e^{2\pi\sqrt{-1}/n}$, and a cyclotomic field $\mathbb{F} = \mathbb{Q}(\zeta)$. Put $\mathbb{E} = \mathbb{R} \cap \mathbb{F} = \mathbb{Q}(\eta)$, where $\eta = \zeta + \zeta^{-1} = 2\cos(2\pi/n)$. Then, we obtain the following exact sequence:

$$1 \longrightarrow \langle\,\bar{}\,\rangle \longrightarrow \mathrm{Gal}(\mathbb{F}/\mathbb{Q}) \longrightarrow \mathrm{Gal}(\mathbb{E}/\mathbb{Q}) \longrightarrow 1$$

where $\bar{}$ means the automorphism induced by the complex conjugation and Gal gives the Galois group (cf [8]). Then, modulo $\langle\,\bar{}\,\rangle$, we can choose m elements $\delta_0 = id, \ldots, \delta_{m-1} \in \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ whose images constitute the whole of $\mathrm{Gal}(\mathbb{E}/\mathbb{Q})$, and we fix them. Let $\mathfrak{O}_{\mathbb{F}} = \mathbb{Z}[\zeta]$, the ring of integers of $\mathbb{F}$, and $\mathfrak{O}_{\mathbb{E}} = \mathbb{Z}[\eta]$, the ring of integers of $\mathbb{E}$.

For a positive real number r and for each $i = 1, \ldots, m-1$, we define a subset

$$\Sigma_i^r = \{\, x \in \mathfrak{O}_{\mathbb{F}} \mid |\delta_i(x)| < r \,\}.$$

Then we define a quasicrystal system by $(\Sigma_1^{r_1}, \Sigma_2^{r_2}, \ldots, \Sigma_{m-1}^{r_{m-1}})$ for any positive real numbers $r_1, r_2, \ldots, r_{m-1}$. For a quasicrystal system $(\Sigma_1^{r_1}, \Sigma_2^{r_2}, \ldots, \Sigma_{m-1}^{r_{m-1}})$, we put, as a realization,

$$\Sigma^{r_1, r_2, \ldots, r_{m-1}} = \bigcap_{i=1}^{m-1} \Sigma_i^{r_i}$$

which is called a quasicrystal associated with $\mathfrak{O}_{\mathbb{F}}$ and $(r_1, r_2, \ldots, r_{m-1})$. Here, to construct our quasicrystals, we selected a special window which is given by $m-1$ parameters $r_1, \ldots, r_{m-1}$. There are many other choices for windows in general (cf [2, 3, 5, 6]). We say that $\Sigma^{r_1, r_2, \ldots, r_{m-1}}$ is isomorphic to $\Sigma^{s_1, s_2, \ldots, s_{m-1}}$ if there exists a $\mathbb{Z}$-linear map ϕ of $\mathfrak{O}_{\mathbb{F}}$ onto $\mathfrak{O}_{\mathbb{F}}$ satisfying

$$\phi(\Sigma_1^{r_1}) = \Sigma_1^{s_1} \qquad \phi(\Sigma_2^{r_2}) = \Sigma_2^{s_2}, \ldots, \phi(\Sigma_{m-1}^{r_{m-1}}) = \Sigma_{m-1}^{s_{m-1}}.$$

Mathematically it is very interesting to study this subset of complex numbers. For $n = 3, 4, 6$, the situation falls into the world of crystals. The most interesting situation is one of the cases when $n = 5, 8, 10, 12$, which implies $\varphi(n) = 4$ and $m = 2$ (cf [1, 4]).

The next one may be the case when $n = 7$, in which case $\varphi(7) = 6$ and $m = 3$. As an example for $\varphi(n) = 8$ and $m = 4$, we will choose $n = 30$.

0305-4470/00/478483+05$30.00

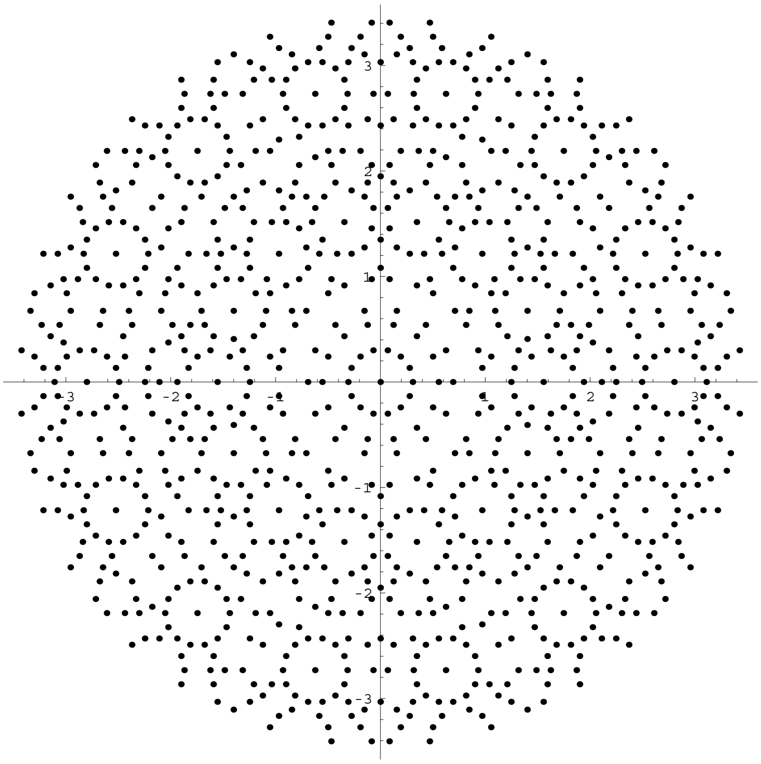

Figure 1. $r_1 = 0.8, r_2 = 52$.

(1) *The case of* $n = 7$. In this case, $m = 3$ and the number of parameters is two. Let $\varepsilon_1 = \eta + 1$ and $\varepsilon_2 = \eta^2 - 1$. Then, the unit group $\mathfrak{O}_{\mathbb{E}}^*$ of $\mathfrak{O}_{\mathbb{E}}$ is

$$\mathfrak{O}_{\mathbb{E}}^* = \{ \pm\varepsilon_1^{k_1}\varepsilon_2^{k_2} \mid k_1, k_2 \in \mathbb{Z} \}.$$

We put $\mathfrak{O}_{\mathbb{E}}^{**} = \{ \varepsilon_1^{2k_1}\varepsilon_2^{2k_2} \mid k_1, k_2 \in \mathbb{Z} \}$, and we choose $\delta_1, \delta_2 \in \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ satisfying

$$\begin{array}{ll} \delta_1(\zeta) = \zeta^2 & \delta_2(\zeta) = \zeta^3 \\ \delta_1(\eta) = \eta^2 - 2 & \delta_2(\eta) = -\eta^2 - \eta + 1 \\ \delta_1(\varepsilon_1) = \varepsilon_2 & \delta_2(\varepsilon_1) = -\varepsilon_1^{-1}\varepsilon_2^{-1} \\ \delta_1(\varepsilon_2) = -\varepsilon_1^{-1}\varepsilon_2^{-1} & \delta_2(\varepsilon_2) = \varepsilon_1. \end{array}$$

Then, using the same argument as in [7], we obtain the following proposition.

Proposition 1. *Two quasicrystals* Σ^{r_1,r_2} *and* Σ^{s_1,s_2} *are isomorphic if and only if there is an element* $\varepsilon \in \mathfrak{O}_{\mathbb{E}}^{**}$ *such that* $(s_i/r_i)^2 = \delta_i(\varepsilon)$ *for* $i = 1, 2$. *That is, two quasicrystals* Σ^{r_1,r_2} *and* Σ^{s_1,s_2} *are isomorphic if and only if the following two integral conditions hold:*

$$\{\log(s_1/r_1)\log\varepsilon_1 + \log(s_2/r_2)\log(\varepsilon_1\varepsilon_2)\}/\{(\log\varepsilon_1)^2 + \log\varepsilon_1\log\varepsilon_2 + (\log\varepsilon_2)^2\} \in \mathbb{Z}$$
$$\{\log(s_1/r_1)\log(\varepsilon_1\varepsilon_2) + \log(s_2/r_2)\log\varepsilon_2\}/\{(\log\varepsilon_1)^2 + \log\varepsilon_1\log\varepsilon_2 + (\log\varepsilon_2)^2\} \in \mathbb{Z}.$$

(2) *The case of* $n = 30$. In this case, $m = 4$ and the number of parameters is three. Let $\varepsilon_1 = \eta$, $\varepsilon_2 = \eta + 1$ and $\varepsilon_3 = \eta^2 - 3$. Then, the unit group $\mathfrak{O}_{\mathbb{E}}^*$ of $\mathfrak{O}_{\mathbb{E}}$ is

$$\mathfrak{O}_{\mathbb{E}}^* = \{\pm\varepsilon_1^{k_1}\varepsilon_2^{k_2}\varepsilon_3^{k_3} \mid k_1, k_2, k_3 \in \mathbb{Z}\}.$$

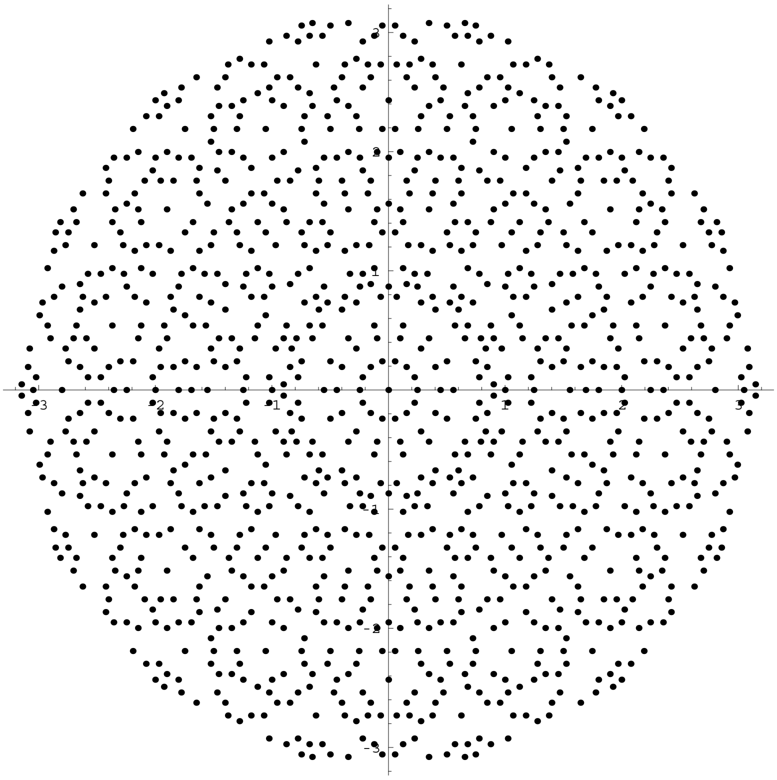

Figure 2. $r_1 = 5, r_2 = 12$.

We put $\mathfrak{O}_{\mathbb{E}}^{**} = \{\varepsilon_1^{2k_1}\varepsilon_2^{2k_2}\varepsilon_3^{2k_3}(\varepsilon_1\varepsilon_2)^{\ell} \mid k_1, k_2, k_3, \ell \in \mathbb{Z}\}$, and we choose $\delta_1, \delta_2, \delta_3 \in \mathrm{Gal}(\mathbb{F}/\mathbb{Q})$ satisfying

$$
\begin{array}{lll}
\delta_1(\zeta) = \zeta^7 & \delta_2(\zeta) = \zeta^{11} & \delta_3(\zeta) = \zeta^{13} \\
\delta_1(\eta) = -\eta^3 + \eta^2 + 3\eta - 2 & \delta_2(\eta) = \eta^3 - 4\eta - 1 & \delta_3(\eta) = -\eta^2 + 2 \\
\delta_1(\varepsilon_1) = \varepsilon_1^{-1}\varepsilon_2^{-1}\varepsilon_3^{-1} & \delta_2(\varepsilon_1) = -\varepsilon_1\varepsilon_3^2 & \delta_3(\varepsilon_1) = -\varepsilon_1^{-1}\varepsilon_2\varepsilon_3^{-1} \\
\delta_1(\varepsilon_2) = \varepsilon_3^{-1} & \delta_2(\varepsilon_2) = -\varepsilon_2^{-1} & \delta_3(\varepsilon_2) = -\varepsilon_3 \\
\delta_1(\varepsilon_3) = -\varepsilon_2 & \delta_2(\varepsilon_3) = -\varepsilon_3^{-1} & \delta_3(\varepsilon_3) = \varepsilon_2^{-1}.
\end{array}
$$

Then, using the same method as in [7], we obtain the following proposition.

Proposition 2. *Two quasicrystals Σ^{r_1,r_2,r_3} and Σ^{s_1,s_2,s_3} are isomorphic if and only if there is an element $\varepsilon \in \mathfrak{O}_{\mathbb{E}}^{**}$ such that $(s_i/r_i)^2 = \delta_i(\varepsilon)$ for $i = 1, 2, 3$. Equivalently, two quasicrystals Σ^{r_1,r_2,r_3} and Σ^{s_1,s_2,s_3} are isomorphic if and only if one of the following two cases occurs.*

Case (a).

$$
\frac{1}{\Delta}\begin{vmatrix} \log(s_1/r_1) & -\log\varepsilon_3 & \log\varepsilon_2 \\ \log(s_2/r_2) & -\log\varepsilon_2 & -\log\varepsilon_3 \\ \log(s_3/r_3) & \log\varepsilon_3 & -\log\varepsilon_2 \end{vmatrix} \in \mathbb{Z}
$$

$$
\frac{1}{\Delta}\begin{vmatrix} -\log(\varepsilon_1\varepsilon_2\varepsilon_3) & \log(s_1/r_1) & \log\varepsilon_2 \\ \log(\varepsilon_1\varepsilon_3^2) & \log(s_2/r_2) & -\log\varepsilon_3 \\ \log(\varepsilon_2/\varepsilon_1\varepsilon_3) & \log(s_3/r_3) & -\log\varepsilon_2 \end{vmatrix} \in \mathbb{Z}
$$

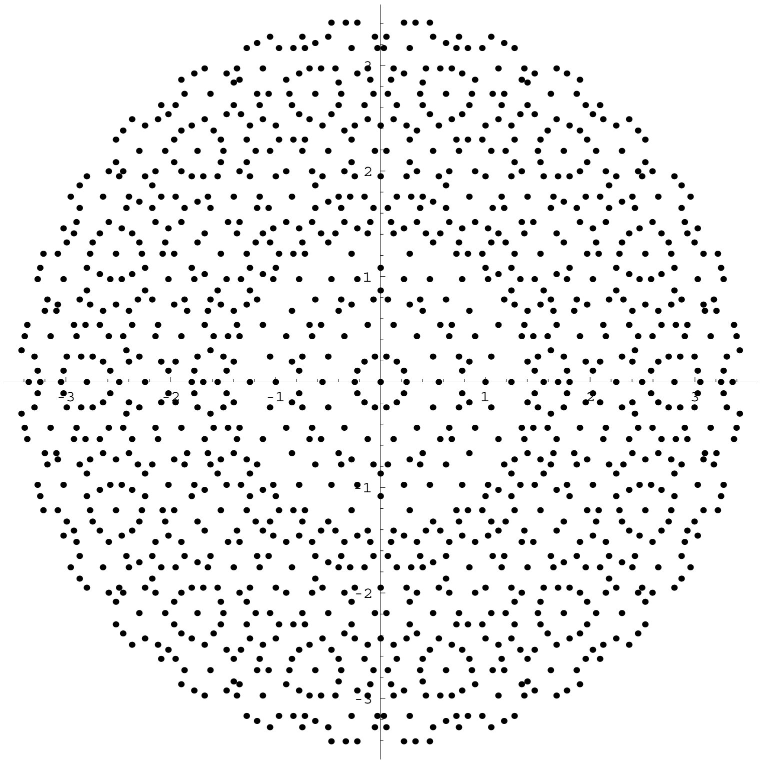

Figure 3. $r_1 = 3, r_2 = 16$.

$$\frac{1}{\Delta}\begin{vmatrix} -\log(\varepsilon_1\varepsilon_2\varepsilon_3) & -\log\varepsilon_3 & \log(s_1/r_1) \\ \log(\varepsilon_1\varepsilon_3^2) & -\log\varepsilon_2 & \log(s_2/r_2) \\ \log(\varepsilon_2/\varepsilon_1\varepsilon_3) & \log\varepsilon_3 & \log(s_3/r_3) \end{vmatrix} \in \mathbb{Z}.$$

Case (b).

$$\frac{1}{\Delta}\begin{vmatrix} \log(s_1\varepsilon_3(\varepsilon_1\varepsilon_2)^{1/2}/r_1) & -\log\varepsilon_3 & \log\varepsilon_2 \\ \log(s_2\varepsilon_2^{1/2}/r_2\varepsilon_3\varepsilon_1^{1/2}) & -\log\varepsilon_2 & -\log\varepsilon_3 \\ \log(s_3\varepsilon_1^{1/2}/r_3\varepsilon_2^{1/2}) & \log\varepsilon_3 & -\log\varepsilon_2 \end{vmatrix} \in \mathbb{Z}$$

$$\frac{1}{\Delta}\begin{vmatrix} -\log(\varepsilon_1\varepsilon_2\varepsilon_3) & \log(s_1\varepsilon_3(\varepsilon_1\varepsilon_2)^{1/2}/r_1) & \log\varepsilon_2 \\ \log(\varepsilon_1\varepsilon_3^2) & \log(s_2\varepsilon_2^{1/2}/r_2\varepsilon_3\varepsilon_1^{1/2}) & -\log\varepsilon_3 \\ \log(\varepsilon_2/\varepsilon_1\varepsilon_3) & \log(s_3\varepsilon_1^{1/2}/r_3\varepsilon_2^{1/2}) & -\log\varepsilon_2 \end{vmatrix} \in \mathbb{Z}$$

$$\frac{1}{\Delta}\begin{vmatrix} -\log(\varepsilon_1\varepsilon_2\varepsilon_3) & -\log\varepsilon_3 & \log(s_1\varepsilon_3(\varepsilon_1\varepsilon_2)^{1/2}/r_1) \\ \log(\varepsilon_1\varepsilon_3^2) & -\log\varepsilon_2 & \log(s_2\varepsilon_2^{1/2}/r_2\varepsilon_3\varepsilon_1^{1/2}) \\ \log(\varepsilon_2/\varepsilon_1\varepsilon_3) & \log\varepsilon_3 & \log(s_3\varepsilon_1^{1/2}/r_3\varepsilon_2^{1/2}) \end{vmatrix} \in \mathbb{Z}$$

where $\Delta = 2\log(\varepsilon_1\varepsilon_3)\{(\log\varepsilon_2)^2 + (\log\varepsilon_3)^2\}$.

Here, | | means the determinant of a matrix. In general, it is possible to try to compute a similar calculation as above for any n, which might be rather complicated sometimes. Figures 1, 2 and 3 are examples for proposition 1.

References

[1] Baake M, Joseph D, Kramer P and Schlottmann M 1990 Root lattices and quasicrystals *J. Phys. A: Math. Gen.* **23** L1037–41

[2] Barache D, Champagne B and Gazeau J-P 1998 Pisot-cyclotomic quasilattices and their symmetry semigroups *Quasicrystals and Discrete Geometry (Fields Institute Monographs 10)* ed J Patera (Providence, RI: American Mathematical Society) pp 15–66

[3] de Bruijn N G 1981 Algebraic theory of Penrose's non-periodic tilings of the plane I *Proc. Konin. Ned. Akad. Weten. Ser. (Indag. Math. 43)* A **84** 39–52

de Bruijn N G 1981 Algebraic theory of Penrose's non-periodic tilings of the plane II *Proc. Konin. Ned. Akad. Weten. Ser. (Indag. Math. 43)* A **84** 53–66

[4] Janot C 1994 *Quasicrystals—A Primer* 2nd edn (Oxford: Oxford Science)

[5] Levine D and Steinhardt P J 1984 A new class of ordered structures *Phys. Rev. Lett.* **53** 2477–80

[6] Levine D and Steinhardt P J 1986 Quasicrystals I: definition and structure *Phys. Rev.* B **34** 596–616

[7] Moody R V and Patera J 1993 Quasicrystals and icosians *J. Phys. A: Math. Gen.* **26** 2829–53

[8] Washington L C 1997 *Introduction to Cyclotomic Fields (Graduate Texts in Mathematics 83)* 2nd edn (New York: Springer)